

Higher-order renormalization of graphene many-body theory

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ABSTRACT: We study the many-body theory of graphene Dirac quasiparticles interacting via the long-range Coulomb potential, taking as a starting point the ladder approximation to different vertex functions. We test in this way the low-energy behavior of the electron system beyond the simple logarithmic dependence of electronic correlators on the high-energy cutoff, which is characteristic of the large- N approximation. We show that the graphene many-body theory is perfectly renormalizable in the ladder approximation, as all higher powers in the cutoff dependence can be absorbed into the redefinition of a finite number of parameters (namely, the Fermi velocity and the weight of the fields) that remain free of infrared divergences even at the charge neutrality point. We illustrate this fact in the case of the vertex for the current density, where a complete cancellation between the cutoff dependences of vertex and electron self-energy corrections becomes crucial for the preservation of the gauge invariance of the theory. The other potentially divergent vertex corresponds to the staggered (sublattice odd) charge density, which is made cutoff independent by a redefinition in the scale of the density operator. This allows to compute a well-defined, scale invariant anomalous dimension to all orders in the ladder series, which becomes singular at a value of the interaction strength marking the onset of chiral symmetry breaking (and gap opening) in the Dirac field theory. The critical coupling we obtain in this way matches with great accuracy the value found with a quite different method, based on the resolution of the gap equation, thus reassuring the predictability of our renormalization approach.

KEYWORDS: renormalization, many-body theory, graphene

1 Introduction

The discovery of graphene, the two-dimensional material made of a one-atom-thick carbon layer[1], has opened new possibilities to investigate fundamental physics as well for devising technological applications. The electron system has relativistic-like invariance at low-energies, mimicking the behavior of Dirac fermions in two spatial dimensions[2–4]. Moreover, the Coulomb repulsion between electrons constitutes the dominant interaction in the graphene layer. This makes the low-energy theory to be a variant of Quantum Electrodynamics, but placed in the strong coupling regime as the ratio of e^2 to the Fermi velocity v_F of the electrons is nominally larger than one.

There have been already several proposals to observe unconventional signatures of the interacting electrons in graphene. It has been for instance remarked that the interaction with impurities carrying a sufficiently large charge should result in anomalous screening properties of the graphene system[5–8]. Furthermore, it was also found long ago that the own e - e interaction in the layer should lead to a linear dependence on energy of the quasiparticle decay rate[9], as a consequence of the vanishing density of states at the charge neutrality point, and in agreement with measurements carried out in graphite[10].

More precisely, it has been shown that the graphene electron system has the properties of a renormalizable quantum field theory, where the parameters flow with the energy scale[11]. In this framework, it was implied that the Fermi velocity should scale logarithmically towards larger values in the low-energy limit, what appears to be confirmed by recent experimental observations in graphene[12].

The graphene electron system is actually an example of electron liquid with strong many-body corrections, which depend significantly on the energy of the interaction processes. In practice, this is manifest in the logarithmic dependence on the high-energy cutoff needed to regularize the contributions to different quantities like the Fermi velocity or the weight of electron quasiparticles. In this type of electron liquid, one has to make sure that these divergences amount to the redefinition of a finite number of parameters in the system. In the context of quantum field theory, this property of renormalizability is crucial to guarantee the predictability of the theory as quantum corrections are taken into account. Otherwise, there is the possibility that the singular dependences on the cutoff cannot be absorbed into the redefinition of a finite number of local operators of the bare theory. This may happen when they take for instance the form of momentum-dependent $\log(|\mathbf{p}|)$ corrections to local operators in the effective action, being then the reflection that the effective low-energy theory cannot be captured in terms of the local fields present in the original model.

At this point, the best evidence of the renormalizability of the model of Dirac fermions in graphene comes from the study of the theory in the limit of large number N of fermion flavors[13], equivalent to the random-phase approximation (RPA). In this regime, it has been shown that all the cutoff dependences of the theory can be absorbed into redefinitions of the Fermi velocity and the weight of the electron quasiparticles, to all orders of the perturbative expansion in e^2/v_F [13] (for other studies of the $1/N$ expansion in graphene, see also Refs. [14] and [15]). Anyhow, many-body corrections only exhibit at large N a

simple logarithmic dependence on the energy cutoff, which makes rather straightforward the renormalization of the theory at this stage.

In this paper we adopt an approach that is opposite in many aspects to that of the large- N expansion, and that is able to probe the structure of the many-body corrections with arbitrary large powers of the cutoff dependence. That is based on the sum of the series of ladder diagrams, that we apply to different interaction vertices of the theory. Within this approach, we will be able to show that the divergent dependences on the cutoff can be reabsorbed in a finite number of parameters of the interacting theory, including the renormalization of the scale of different bilinears of the Dirac fermion fields. These renormalized quantities will prove to be independent of any infrared scale (Fermi energy, external momenta), making the low-energy limit of the many-body theory perfectly well-defined even at the charge neutrality point.

From a practical point of view, the motivation for focusing on the sum of the ladder series lies in that it encodes the most divergent diagrams at each level of the perturbative expansion for the undoped electron system[16]. This makes highly nontrivial the process of renormalization, by which one has to remove in general divergent corrections that behave like the n -th power of the logarithm of the cutoff, when looking at the n -th perturbative level. In practice, we will illustrate the usefulness of the renormalization approach in the computation of observables like the anomalous dimensions of composite operators, which become determined just by the value of the renormalized coupling constant. This will allow us to address in particular the question of the dynamical breakdown of the chiral symmetry in the electron system[17–29], which can be characterized in terms of the singular behavior of the corresponding anomalous dimension at a certain critical value of the coupling constant[30, 31].

2 Dirac many-body theory

Graphene is a 2D crystal of carbon atoms forming a honeycomb lattice, such that its low-energy electron quasiparticles are disposed into conical conduction and valence bands that touch at the six corners of the Brillouin zone[4]. Of all six Fermi points, there are only two independent classes of electronic excitations. Thus, the low-energy electronic states can be encoded into a set of four-dimensional Dirac spinors $\{\psi_i\}$, which are characterized by having linear energy-momentum dispersion $\varepsilon(\mathbf{p}) = v_F|\mathbf{p}|$. The index i accounts for the two spin degrees of freedom, but may also allow to extend formally the analysis for a higher number N of Dirac spinors. The kinetic term of the hamiltonian in this low-energy theory is given by

$$H_0 = -iv_F \int d^2r \bar{\psi}_i(\mathbf{r}) \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \psi_i(\mathbf{r}) \quad (2.1)$$

where $\bar{\psi}_i = \psi_i^\dagger \gamma_0$ and $\{\gamma_\sigma\}$ is a collection of four-dimensional matrices such that $\{\gamma_\mu, \gamma_\nu\} = 2 \text{diag}(1, -1, -1)$. They can be conveniently represented in terms of Pauli matrices as $\gamma_{0,1,2} = (\sigma_3, \sigma_3\sigma_1, \sigma_3\sigma_2) \otimes \sigma_3$, where the first factor acts on the two sublattice components of the honeycomb lattice and the second factor operates on the set of two independent Fermi points.

In this paper we focus on the effects of the long-range Coulomb interaction in the graphene electron system. The density of states vanishes at the Fermi points connecting the conduction and valence bands, so that a sensible starting point for the e - e interaction is given by the unscreened potential $V(\mathbf{r}) = e^2/4\pi|\mathbf{r}|$. The long-range Coulomb repulsion governs actually the properties of the electron system at low energies, since it is the only interaction that is not suppressed, at the classical level, when scaling the many-body theory in the limit of very large distances. If we add to (2.1) the contribution from the Coulomb interaction, we get the expression of the full hamiltonian

$$H = -iv_F \int d^2r \bar{\psi}_i(\mathbf{r}) \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \psi_i(\mathbf{r}) + \frac{e^2}{8\pi} \int d^2r_1 \int d^2r_2 \rho(\mathbf{r}_1) \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} \rho(\mathbf{r}_2) \quad (2.2)$$

with $\rho(\mathbf{r}) = \bar{\psi}_i(\mathbf{r}) \gamma_0 \psi_i(\mathbf{r})$. The total action of the system is

$$S = \int dt \int d^2r \bar{\psi}_i(\mathbf{r}) (i\gamma_0 \partial_t + iv_F \boldsymbol{\gamma} \cdot \boldsymbol{\nabla}) \psi_i(\mathbf{r}) - \frac{e^2}{8\pi} \int dt \int d^2r_1 \int d^2r_2 \rho(\mathbf{r}_1) \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} \rho(\mathbf{r}_2) \quad (2.3)$$

This action is invariant under the combined transformation of the space and time variables and the scale of the fields

$$t' = st, \quad \mathbf{r}' = s\mathbf{r}, \quad \psi'_i = s^{-1}\psi_i \quad (2.4)$$

This means in particular that the strength of the interaction is not diminished (contrary to the case of a short-range interaction) when zooming into the low-energy limit $s \rightarrow \infty$.

This analysis shows that the Coulomb repulsion mediated by the long-distance $1/|\mathbf{r}|$ potential is the only interaction that may prevail in the low-energy regime of the electron system. It is clear that any other e - e interaction without the $1/|\mathbf{r}|$ tail, as those that arise effectively from phonon exchange, will be suppressed at least by a power of $1/s$ under the change of variables (2.4). The Dirac field theory with long-range Coulomb interaction has indeed the property of being scale invariant at this classical level, which provides a good starting point to investigate the behavior of the many-body corrections upon scale transformations towards the long-wavelength limit $s \rightarrow \infty$.

In fact, the many-body theory does not preserve in general the scale invariance of the classical action (2.3), as a high-energy cutoff Λ has to be introduced to obtain finite results in the computation of many-body corrections to different observables. The analysis of the cutoff dependence of the many-body theory provides deeper insight into the effective low-energy theory. If the theory is renormalizable, it must be possible to absorb all powers of the cutoff dependence into a redefinition of the parameters in the action (2.3). This should be therefore modified to read

$$S = Z_{\text{kin}} \int dt \int d^2r \bar{\psi}_i(\mathbf{r}) (i\gamma_0 \partial_t + iZ_v v_F \boldsymbol{\gamma} \cdot \boldsymbol{\nabla}) \psi_i(\mathbf{r}) - Z_{\text{int}} \frac{e^2}{8\pi} \int dt \int d^2r_1 \int d^2r_2 \rho(\mathbf{r}_1) \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} \rho(\mathbf{r}_2) \quad (2.5)$$

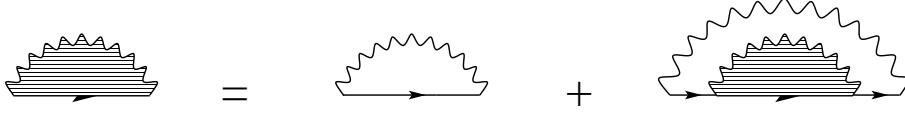


Figure 1. Diagrammatic representation of the ladder approximation for the electron self-energy.

The assumption is that Z_{kin} , Z_v and Z_{int} (and other renormalization factors for composite operators that do not appear in (2.3)) can only depend on the cutoff, while they must be precisely chosen to render all electronic correlators cutoff independent.

The renormalizability of the graphene Dirac field theory is a nontrivial statement, since it amounts to the fact that all the many-body corrections depending on the high-energy cutoff must reproduce the structure of the simple local operators that appear in (2.3). It turns out for instance that many of the individual contributions to a given correlator have dependences in momentum space of the form $\log^n(|\mathbf{p}|) \log^m(\Lambda)$. These are nonlocal corrections that cannot be reabsorbed into the action (2.5), and the fact that all these nonlocal terms cancel out in the final result for a correlator is a remarkable property of a renormalizable theory. Non-renormalizable theories have in this regard an essential lack of predictability, as $\log^n(|\mathbf{p}|)$ corrections do not make viable the characterization of the low-energy effective theory in terms of a few local operators, which may be in turn the reflection that it is not actually captured by the original fields formulated in the many-body theory.

3 Electron self-energy and Fermi velocity renormalization

We first consider the cutoff dependence of the electron self-energy in the ladder approximation. We define this approach in terms of the self-consistent equation represented in Fig. 1. Diagrammatically, it corresponds to build the electron self-energy by iteration in the number of “rainbow-like” interactions between the Dirac fermion lines. A similar approach will be used afterwards to define the ladder approximation for the vertices of the charge and current density operators.

Before dealing with the actual ladder series, we establish our representation of the free propagators by describing the computation of the lowest-order self-energy diagram. The free propagation of the Dirac fermions corresponds for instance to the expectation value

$$\begin{aligned} \langle \psi_i(\mathbf{k}, \omega) \bar{\psi}_i(\mathbf{k}, \omega) \rangle_{\text{free}} &= iG_0(\mathbf{k}, \omega) \\ &= i \frac{-\gamma_0 \omega + v_F \boldsymbol{\gamma} \cdot \mathbf{k}}{-\omega^2 + v_F^2 \mathbf{k}^2 - i\eta} \end{aligned} \quad (3.1)$$

On the other hand, the interaction lines stand in momentum space for the product of $-i$ times the Fourier transform of the Coulomb potential, that turns out to be in two spatial dimensions

$$V(\mathbf{q}) = \frac{2\pi}{|\mathbf{q}|} \quad (3.2)$$

The first-order electron self-energy diagram, that we will denote by $\Sigma_1(\mathbf{k})$, needs to be regularized by introducing a high-energy cutoff Λ in the momentum integrals. We have actually

$$i\Sigma_1(\mathbf{k}) = -\frac{e^2}{2} \int \frac{d^2p}{(2\pi)^2} \int \frac{d\omega_p}{2\pi} \gamma_0 \frac{-\gamma_0\omega_p + v_F\boldsymbol{\gamma} \cdot \mathbf{p}}{-\omega_p^2 + v_F^2\mathbf{p}^2 - i\eta} \gamma_0 \frac{1}{|\mathbf{k} - \mathbf{p}|} \quad (3.3)$$

which leads to a contribution proportional to $\boldsymbol{\gamma} \cdot \mathbf{k}$ that must be logarithmically divergent by simple dimensional counting. If we bound the integration in momentum space such that $v_F|\mathbf{p}| < \Lambda$, we get the result

$$\Sigma_1(\mathbf{k}) \approx \frac{e^2}{16\pi} \boldsymbol{\gamma} \cdot \mathbf{k} \log \Lambda \quad (3.4)$$

that corresponds to the well-known renormalization of the Fermi velocity by the Coulomb interaction in the Dirac many-body theory[11, 13].

From now on, we will choose a convenient regularization method to compute the divergent as well as the finite corrections to electronic correlators at each perturbative level. That consists in the analytic continuation in the number of space dimensions[32], by which the momentum integrals are computed at dimension $D = 2 - \epsilon$ [11]. With this method, dependences on $\log \Lambda$ are traded in general by $1/\epsilon$ poles. In the above instance of the electron self-energy, we get after integration over ω_p

$$i\Sigma_1(\mathbf{k}) = i\frac{e_0^2}{4} \int \frac{d^Dp}{(2\pi)^D} \boldsymbol{\gamma} \cdot \mathbf{p} \frac{1}{|\mathbf{p}|} \frac{1}{|\mathbf{k} - \mathbf{p}|} \quad (3.5)$$

where e_0 is a parameter whose dimensions are given by an auxiliary momentum scale μ through the relation

$$e_0 = \mu^{\epsilon/2} e \quad (3.6)$$

The calculation then proceeds as follows:

$$\begin{aligned} \Sigma_1(\mathbf{k}) &= \frac{e_0^2}{4\pi} \int \frac{d^Dp}{(2\pi)^D} \boldsymbol{\gamma} \cdot \mathbf{p} \int_0^1 dx \frac{x^{-1/2}(1-x)^{-1/2}}{(\mathbf{k} - \mathbf{p})^2 x + \mathbf{p}^2(1-x)} \\ &= \frac{e_0^2}{4\pi} \int \frac{d^Dp}{(2\pi)^D} \boldsymbol{\gamma} \cdot \mathbf{k} \int_0^1 dx \frac{x^{1/2}(1-x)^{-1/2}}{\mathbf{p}^2 + \mathbf{k}^2 x(1-x)} \\ &= \frac{e_0^2}{4\pi} \boldsymbol{\gamma} \cdot \mathbf{k} \int_0^1 dx \frac{\sqrt{x}}{\sqrt{1-x}} \frac{\Gamma(1-D/2)}{(4\pi)^{D/2}} \frac{1}{(\mathbf{k}^2 x(1-x))^{1-D/2}} \\ &= \frac{e_0^2}{(4\pi)^2} \boldsymbol{\gamma} \cdot \mathbf{k} \frac{(4\pi)^{\epsilon/2} \Gamma(\frac{1}{2}\epsilon) \Gamma(\frac{3-\epsilon}{2}) \Gamma(\frac{1-\epsilon}{2})}{|\mathbf{k}|^\epsilon \Gamma(2-\epsilon)} \end{aligned} \quad (3.7)$$

From the latter expression we find the pole as $\epsilon \rightarrow 0$

$$\Sigma_1(\mathbf{k}) \approx \frac{e^2}{16\pi} \boldsymbol{\gamma} \cdot \mathbf{k} \frac{1}{\epsilon} \quad (3.8)$$

We are anyhow interested in the result of computing the electron self-energy in the ladder approximation defined in Fig. 1. It is easily realized that the solution $\Sigma_{\text{ladder}}(\mathbf{k})$ of the self-consistent equation must have the structure

$$\Sigma_{\text{ladder}}(\mathbf{k}) = f(\mathbf{k}) \boldsymbol{\gamma} \cdot \mathbf{k} \quad (3.9)$$

with a scalar function $f(\mathbf{k})$. The self-energy then satisfies

$$i\Sigma_{\text{ladder}}(\mathbf{k}) = i\Sigma_1(\mathbf{k}) + \frac{e_0^2}{2} \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega_p}{2\pi} \Sigma_{\text{ladder}}(\mathbf{p}) \frac{\omega_p^2 + v_F^2 \mathbf{p}^2}{(-\omega_p^2 + v_F^2 \mathbf{p}^2 - i\eta)^2} \frac{1}{|\mathbf{k} - \mathbf{p}|} \quad (3.10)$$

The solution of Eq. (3.10) reflects a particular feature of the graphene many-body theory in the static limit (i.e. when the effective interaction is supposed to be frequency independent). It can be checked that the second term in the self-consistent equation identically vanishes, as a result of performing the integration over the frequency variable, and irrespective of the actual momentum dependence of $\Sigma_{\text{ladder}}(\mathbf{k})$. We have indeed, by performing a Wick rotation to imaginary frequency $\bar{\omega}_k = -i\omega_k$,

$$\int \frac{d\omega_p}{2\pi} \frac{\omega_p^2 + v_F^2 \mathbf{p}^2}{(-\omega_p^2 + v_F^2 \mathbf{p}^2 - i\eta)^2} = i \int \frac{d\bar{\omega}_p}{2\pi} \frac{-\bar{\omega}_p^2 + v_F^2 \mathbf{p}^2}{(\bar{\omega}_p^2 + v_F^2 \mathbf{p}^2)^2} = 0 \quad (3.11)$$

This result implies that the solution of Eq. (3.10) must coincide with the first-order contribution

$$\Sigma_{\text{ladder}}(\mathbf{k}) = \Sigma_1(\mathbf{k}) \quad (3.12)$$

This vanishing of higher-order corrections to the electron self-energy in the ladder approximation can be actually seen as the consequence of a wider symmetry operating in the graphene many-body theory. We can extend the sum of self-energy diagrams in the ladder series to include contributions where the electron lines in the ladder diagrams are corrected by the own electron self-energy. This leads to the sum of a much broader class of diagrams, that are encoded in the self-consistent exchange approximation depicted in Fig. 2. If we represent the electron self-energy in this approach by

$$\Sigma_{\text{SCEX}}(\mathbf{k}) = \tilde{f}(\mathbf{k}) \boldsymbol{\gamma} \cdot \mathbf{k} \quad (3.13)$$

the self-consistent equation can be written as

$$i\Sigma_{\text{SCEX}}(\mathbf{k}) = -\frac{e_0^2}{2} \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega_p}{2\pi} \gamma_0 \frac{-\gamma_0 \omega_p + (v_F + \tilde{f}(\mathbf{p})) \boldsymbol{\gamma} \cdot \mathbf{p}}{-\omega_p^2 + (v_F + \tilde{f}(\mathbf{p}))^2 \mathbf{p}^2 - i\eta} \gamma_0 \frac{1}{|\mathbf{k} - \mathbf{p}|} \quad (3.14)$$

The key observation is that, for the same reason that the final expression for the first-order self-energy (3.3) does not depend on the Fermi velocity v_F , the integral in Eq. (3.14) turns out to be independent of the function $\tilde{f}(\mathbf{k})$. One can for instance redefine the scale of the frequency from ω_p to $(1 + \tilde{f}(\mathbf{p})/v_F)\omega_p$, in such a way that the integrand in Eq. (3.14) falls into the corresponding first-order expression in (3.3). This proves that, also in the more comprehensive self-consistent exchange approximation, the electron self-energy coincides with the first-order result

$$\Sigma_{\text{SCEX}}(\mathbf{k}) = \Sigma_1(\mathbf{k}) \quad (3.15)$$

In its simplicity, the result expressed in Eq. (3.15) accounts for the vanishing of a vast class of corrections to the electron self-energy in graphene. It can be interpreted as a kind of no-renormalization theorem that protects the Fermi velocity from being modified

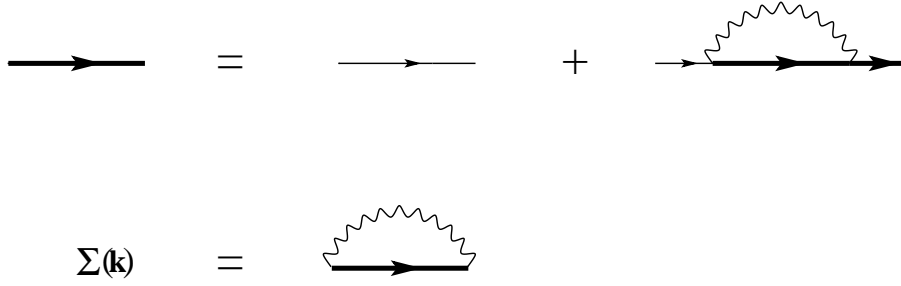


Figure 2. Diagrammatic representation of the self-consistent exchange approximation for the electron self-energy.

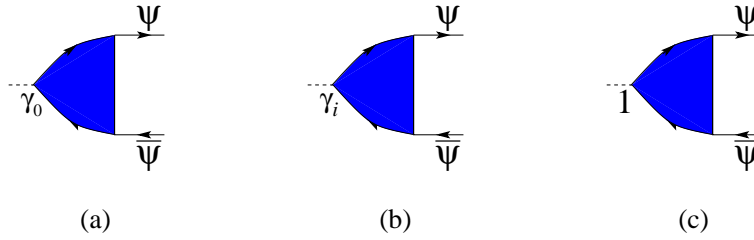


Figure 3. Diagrammatic representation of the vertices for (a) total charge density, (b) current density, and (c) staggered charge density.

by higher-order effects, which remains valid under the assumption of static (frequency-independent) screening of the Coulomb interaction. As we will see, this translates into a remarkable cancellation of corrections to the vertex of the current density operator, in a nontrivial check of a gauge invariance that is hidden in the original formulation of the theory.

4 Charge and current density correlations

We study next the way in which many-body corrections dress the different interaction vertices in graphene. This includes the inspection of the own Coulomb interaction, that we analyze by looking for corrections of the coupling to the total charge density. The Dirac field theory allows anyhow for the consideration of more general vertices that take into account the pseudospin current γ and the spinor structure of the fermion fields. We will pay attention in what follows to the vertices for the total charge, the pseudospin current, and the staggered (sublattice odd) charge density, which are represented in Fig. 3.

The analysis of the vertices for the total charge and the pseudospin current is particularly relevant since, if we think of them as operators that may be switched on in the action of the electron system, it becomes clear that they should be related by gauge invariance to

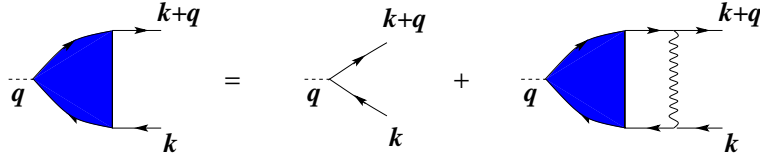


Figure 4. Self-consistent diagrammatic equation for a generic vertex Γ_i in the ladder approximation.

the terms in the kinetic action. That is, we can start with an extended action given by

$$S = Z_{\text{kin}} \int dt \int d^2r \bar{\psi}_i(\mathbf{r})(i\gamma_0\partial_t + iZ_v v_F \boldsymbol{\gamma} \cdot \boldsymbol{\nabla})\psi_i(\mathbf{r}) + e \int dt \int d^2r \bar{\psi}_i(\mathbf{r})(Z'_{\text{int}}\gamma_0 A_0 + Z''_{\text{int}}\boldsymbol{\gamma} \cdot \mathbf{A})\psi_i(\mathbf{r}) \quad (4.1)$$

where A_0 and \mathbf{A} play the role of auxiliary fields mediating the interactions of the total charge and the pseudospin current. A gauge transformation of the Dirac fields

$$\tilde{\psi}_i(\mathbf{r}) = e^{ie\theta(\mathbf{r},t)}\psi_i(\mathbf{r}) \quad (4.2)$$

amounts to a shift of the auxiliary fields A_0 and \mathbf{A} . Thus, the invariance of the many-body theory under (4.2) can be tested by checking that the renormalization factors Z'_{int} and Z''_{int} match with the respective factors from the renormalization of the electron self-energy.

This question of the gauge invariance is an interesting point regarding the graphene many-body theory, as it was shown long ago that the four-fermion interactions in the graphene electron system can be obtained from a suitable projection of the full relativistic interaction mediated by photons in three spatial dimensions[11]. It has been actually proven that the renormalization of the theory, when carried out to first order in perturbation theory, is consistent with the above mentioned gauge invariance. In the present instance, we will also use the renormalization properties of the vertices to check the underlying gauge invariance to higher orders in the ladder approximation supplemented by electron self-energy corrections.

As in the case of the electron self-energy, we define the ladder approach for the vertices by means of a self-consistent equation, represented now in Fig. 4. In principle, one can solve the equation by means of an iterative procedure, ending up with the equivalent of a ladder series for the different vertices. We will then improve this diagrammatic approach in a second stage, by assuming that the internal fermion lines in the self-consistent equation are themselves corrected by the electron self-energy, which will prove to be crucial to preserve the gauge invariance of the theory.

4.1 Charge density vertex

We define the vertex for the total charge density in frequency and momentum space as

$$\Gamma_0(\mathbf{q}, \omega_q; \mathbf{k}, \omega_k) = \langle \rho(\mathbf{q}, \omega_q) \psi_i(\mathbf{k} + \mathbf{q}, \omega_k + \omega_q) \bar{\psi}_i(\mathbf{k}, \omega_k) \rangle_{\text{1PI}} \quad (4.3)$$

where ρ is given in real space by

$$\rho(\mathbf{r}) = \bar{\psi}_i(\mathbf{r}) \gamma_0 \psi_i(\mathbf{r}) \quad (4.4)$$

and the right-hand-side of (4.3) is computed by considering only the one-particle-irreducible vertex diagrams. In this way, the possible renormalization required to render Γ_0 cutoff independent should amount to a simple multiplication of the vertex by a factor, that is the same Z'_{int} appearing in (4.1).

The vertex Γ_0 is a dimensionless quantity, which means that, in order to isolate the singular dependence on the cutoff, it is enough to study the limit $\mathbf{q} \rightarrow 0$ and $\omega_q \rightarrow 0$. Then, the self-consistent equation represented in Fig. 4 becomes

$$\Gamma_0(\mathbf{0}, 0; \mathbf{k}, \omega_k) = \gamma_0 + i \frac{e_0^2}{2} \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_p}{2\pi} \Gamma_0(\mathbf{0}, 0; \mathbf{p}, \omega_p) \frac{\omega_p^2 + v_F^2 \mathbf{p}^2}{(-\omega_p^2 + v_F^2 \mathbf{p}^2 - i\eta)^2} \frac{1}{|\mathbf{k} - \mathbf{p}|} \quad (4.5)$$

It is clear that the solution of (4.5) cannot depend on the frequency ω_k of the external fermion lines. Therefore, the integral at the right-hand-side of the equation must be identically zero, for the same reason that the integral in Eq. (3.10) was also vanishing. This means that the vertex Γ_0 is independent of the cutoff in the ladder approximation. Repeating here the argument at the end of the last section, it turns out that the same statement holds true even when the fermion propagators in (4.5) are corrected with the electron self-energy (3.7). Again, the integral in (4.5) vanishes irrespective of the momentum-dependent corrections to v_F , leaving Γ_0 cutoff independent in this approach.

The cutoff independence of Γ_0 agrees with the absence of wavefunction renormalization ($Z_{\text{kin}} = 1$) in the self-consistent exchange approximation applied to the electron self-energy. The trivial result

$$Z_{\text{kin}} = Z'_{\text{int}} = 1 \quad (4.6)$$

is the first check of the gauge invariance of the theory. The vanishing of the many-body corrections lies in this instance in the particular structure of the ladder approximation and, in this regard, it is a result that holds even after dressing the interaction with the static (frequency-independent) RPA screening of the Coulomb potential.

4.2 Current density vertex

The irreducible vertex for the current density is defined in this case by

$$\Gamma_c(\mathbf{q}, \omega_q; \mathbf{k}, \omega_k) = \langle \rho_c(\mathbf{q}, \omega_q) \psi_i(\mathbf{k} + \mathbf{q}, \omega_k + \omega_q) \bar{\psi}_i(\mathbf{k}, \omega_k) \rangle_{\text{1PI}} \quad (4.7)$$

where the current density operator is given in real space by

$$\rho_c(\mathbf{r}) = \bar{\psi}_i(\mathbf{r}) \boldsymbol{\gamma} \psi_i(\mathbf{r}) \quad (4.8)$$

We anticipate the fact that the computation of the vertex may give rise to dependences on the high-energy cutoff, that are supposed to be absorbed in the renormalization factor Z''_{int} .

The vertex Γ_c is a two-dimensional vector, but its analysis can be greatly simplified by considering as before the limit $\mathbf{q} \rightarrow 0$ and $\omega_q \rightarrow 0$. The self-consistent equation depicted in Fig. 4 takes then the form

$$\Gamma_c(\mathbf{0}, 0; \mathbf{k}, \omega_k) = \gamma + i \frac{e_0^2}{2} \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_p}{2\pi} \gamma_0 \frac{-\gamma_0 \omega_p + v_F \gamma \cdot \mathbf{p}}{-\omega_p^2 + v_F^2 \mathbf{p}^2 - i\eta} \Gamma_c(\mathbf{0}, 0; \mathbf{p}, \omega_p) \frac{-\gamma_0 \omega_p + v_F \gamma \cdot \mathbf{p}}{-\omega_p^2 + v_F^2 \mathbf{p}^2 - i\eta} \gamma_0 \frac{1}{|\mathbf{k} - \mathbf{p}|} \quad (4.9)$$

We resort at this point to an iterative resolution of (4.9), by which we can obtain a recursion between consecutive orders in the expansion of the vertex in powers of the interaction strength. This procedure shows that $\Gamma_c(\mathbf{0}, 0; \mathbf{k}, \omega_k)$ has a part proportional to γ and another contribution proportional to $\mathbf{k}(\gamma \cdot \mathbf{k})$. From dimensional arguments, one can see that the solution of (4.9) must take the form

$$\Gamma_c(\mathbf{0}, 0; \mathbf{k}, \omega_k) = \gamma \left(1 + \sum_{n=1}^{\infty} \lambda_0^n \frac{r_n}{|\mathbf{k}|^{n\epsilon}} \right) + \mathbf{n}_k (\gamma \cdot \mathbf{n}_k) \sum_{n=1}^{\infty} \lambda_0^n \frac{r'_n}{|\mathbf{k}|^{n\epsilon}} \quad (4.10)$$

where we have called $\mathbf{n}_k = \mathbf{k}/|\mathbf{k}|$ and

$$\lambda_0 = \frac{e_0^2}{4\pi v_F} \quad (4.11)$$

If we insert for instance a given order of the expansion with coefficient r_n inside the integral in Eq. (4.9), we get

$$\begin{aligned} & -\frac{e_0^2}{2} \int \frac{d^D p}{(2\pi)^D} \frac{d\bar{\omega}_p}{2\pi} \gamma_0 \frac{-i\gamma_0 \bar{\omega}_p + v_F \gamma \cdot \mathbf{p}}{\bar{\omega}_p^2 + v_F^2 \mathbf{p}^2} \gamma \frac{r_n}{|\mathbf{p}|^{n\epsilon}} \frac{-i\gamma_0 \bar{\omega}_p + v_F \gamma \cdot \mathbf{p}}{\bar{\omega}_p^2 + v_F^2 \mathbf{p}^2} \gamma_0 \frac{1}{|\mathbf{k} - \mathbf{p}|} \\ & = r_n \frac{e_0^2}{4v_F} \int \frac{d^D p}{(2\pi)^D} \left(\gamma \frac{1}{|\mathbf{p}|^{1+n\epsilon}} - \mathbf{p}(\gamma \cdot \mathbf{p}) \frac{1}{|\mathbf{p}|^{3+n\epsilon}} \right) \frac{1}{|\mathbf{k} - \mathbf{p}|} \\ & = r_n \frac{e_0^2}{4v_F} \int \frac{d^D p}{(2\pi)^D} (\gamma - \mathbf{n}_p(\gamma \cdot \mathbf{n}_p)) \frac{\Gamma(1 + \frac{n\epsilon}{2})}{\sqrt{\pi} \Gamma(\frac{1+n\epsilon}{2})} \int_0^1 \frac{x^{-1/2} (1-x)^{-(1-n\epsilon)/2}}{((\mathbf{k} - \mathbf{p})^2 x + \mathbf{p}^2 (1-x))^{1+n\epsilon/2}} \\ & = \gamma \lambda_0 \frac{r_n}{|\mathbf{k}|^{(n+1)\epsilon}} \frac{(4\pi)^{\epsilon/2}}{4} \left(\frac{\Gamma(\frac{n+1}{2}\epsilon) \Gamma(\frac{1-(n+1)\epsilon}{2}) \Gamma(\frac{1-\epsilon}{2})}{\sqrt{\pi} \Gamma(\frac{1+n\epsilon}{2}) \Gamma(1 - \frac{(n+2)\epsilon}{2})} - \frac{\Gamma(\frac{n+1}{2}\epsilon) \Gamma(\frac{1-(n+1)\epsilon}{2}) \Gamma(\frac{3-\epsilon}{2})}{2\sqrt{\pi} \Gamma(\frac{3+n\epsilon}{2}) \Gamma(2 - \frac{(n+2)\epsilon}{2})} \right) \\ & \quad - \mathbf{n}_k(\gamma \cdot \mathbf{n}_k) \lambda_0 \frac{r_n}{|\mathbf{k}|^{(n+1)\epsilon}} \frac{(4\pi)^{\epsilon/2}}{4} \frac{\Gamma(1 + \frac{n+1}{2}\epsilon) \Gamma(\frac{3-(n+1)\epsilon}{2}) \Gamma(\frac{1-\epsilon}{2})}{\sqrt{\pi} \Gamma(\frac{3+n\epsilon}{2}) \Gamma(2 - \frac{(n+2)\epsilon}{2})} \end{aligned} \quad (4.12)$$

On the other hand, by inserting any term of the expansion (4.10) with r'_n coefficient inside the integral of the self-consistent equation, we get always a vanishing result due to Eq. (3.11). We obtain therefore the recurrence relations

$$r_{n+1} = \frac{(4\pi)^{\epsilon/2}}{4} \left(\frac{\Gamma(\frac{n+1}{2}\epsilon) \Gamma(\frac{1-(n+1)\epsilon}{2}) \Gamma(\frac{1-\epsilon}{2})}{\sqrt{\pi} \Gamma(\frac{1+n\epsilon}{2}) \Gamma(1 - \frac{(n+2)\epsilon}{2})} - \frac{\Gamma(\frac{n+1}{2}\epsilon) \Gamma(\frac{1-(n+1)\epsilon}{2}) \Gamma(\frac{3-\epsilon}{2})}{2\sqrt{\pi} \Gamma(\frac{3+n\epsilon}{2}) \Gamma(2 - \frac{(n+2)\epsilon}{2})} \right) r_n \quad (4.13)$$

$$r'_{n+1} = -\frac{(4\pi)^{\epsilon/2} \Gamma\left(1 + \frac{n+1}{2}\epsilon\right) \Gamma\left(\frac{3-(n+1)\epsilon}{2}\right) \Gamma\left(\frac{1-\epsilon}{2}\right)}{4 \sqrt{\pi} \Gamma\left(\frac{3+n\epsilon}{2}\right) \Gamma\left(2 - \frac{(n+2)\epsilon}{2}\right)} r_n \quad (4.14)$$

We observe from (4.13) that the expansion of the vertex develops increasing divergences in the cutoff, that manifest as poles in the limit $\epsilon \rightarrow 0$. The key point is whether these divergences can be absorbed by a suitable renormalization factor. The current density $\boldsymbol{\rho}_c(\mathbf{r})$ is not an elementary field of the many-body theory, which means that its correlators need to be renormalized by appropriate rescaling of the own current density. Alternatively, if we include the composite field with its own coupling in the action (4.1), it is the renormalization factor Z''_{int} which needs to be adjusted to render the correlators cutoff independent. Then, a renormalized vertex $\boldsymbol{\Gamma}_{c,\text{ren}}$, finite in the limit $\epsilon \rightarrow 0$, has to be obtained by the multiplicative renormalization

$$\boldsymbol{\Gamma}_{c,\text{ren}} = Z''_{\text{int}} \boldsymbol{\Gamma}_c \quad (4.15)$$

The renormalization factor may have in general the structure

$$Z''_{\text{int}} = 1 + \sum_{i=1}^{\infty} \frac{c_i(\lambda)}{\epsilon^i} \quad (4.16)$$

in terms of the dimensionless physical coupling

$$\lambda = \frac{e^2}{4\pi v_F} \quad (4.17)$$

It is a nontrivial fact that all the poles in $\boldsymbol{\Gamma}_c$ may be canceled against multiplication by Z''_{int} , allowing only for the dependence of the coefficients c_i on the coupling constant. We have checked that this is indeed the case, up to the order λ^{18} we have been able to carry out the numerical computation of the vertex. We have found for instance for the first terms in the expansion (4.16)

$$\begin{aligned} c_1(\lambda) = & -\frac{1}{4}\lambda - \frac{1}{64}(1 + \log(16))\lambda^2 - \frac{1}{384}(-1 + 3\log^2(4) + \log(64))\lambda^3 \\ & - \frac{-9 - 72\log(2) + 384\log^2(2) + 128\log^2(2)\log(16) + 12\zeta(3)}{24576}\lambda^4 \\ & - \frac{3 - 30\log(2) - 60\log^2(2) + 400\log^3(2) + 400\log^4(2) + 6\zeta(3) + 6\log(16)\zeta(3)}{24576}\lambda^5 \\ & + \dots \end{aligned} \quad (4.18)$$

$$\begin{aligned} c_2(\lambda) = & \frac{1}{32}\lambda^2 + \frac{1}{256}(1 + 4\log(2))\lambda^3 - \frac{13 - 120\log(2) - 120\log(2)\log(4)}{24576}\lambda^4 \\ & - \frac{13 + 64\log(2) - 528\log^2(2) - 176\log^2(2)\log(16) - 12\zeta(3)}{98304}\lambda^5 + \dots \end{aligned} \quad (4.19)$$

$$c_3(\lambda) = -\frac{1}{384}\lambda^3 - \frac{1 + 4\log(2)}{2048}\lambda^4 - \frac{-5 + 72\log(2) + 144\log^2(2)}{98304}\lambda^5 + \dots \quad (4.20)$$

$$c_4(\lambda) = \frac{1}{6144}\lambda^4 + \frac{1 + 4\log(2)}{24576}\lambda^5 + \dots \quad (4.21)$$

$$c_5(\lambda) = -\frac{1}{122880}\lambda^5 + \dots \quad (4.22)$$

The important point about this result for Z''_{int} is that it does not depend on the momenta of the vertex Γ_c . This means that it represents a local divergence as $\epsilon \rightarrow 0$, and it can be therefore understood as the renormalization of a local operator in the action (4.1).

Another important consequence of the actual expression of the functions $c_i(\lambda)$ is that observable quantities derived from Z''_{int} , as for instance the anomalous dimension γ_c of the current operator, turn out to be finite in the limit $\epsilon \rightarrow 0$. The dimension γ_c measures in particular the anomalous scaling of ρ_c under changes in the units of energy and momentum in the system. The vertex Γ_c is formally a dimensionless quantity, but the renormalization process introduces scale dependence on the auxiliary momentum μ , in such a way that

$$\Gamma_{c,\text{ren}} \sim \mu^{\gamma_c} \quad (4.23)$$

The anomalous scaling of the vertex comes only from the dependence of the renormalization factor Z''_{int} on the μ scale, so that

$$\gamma_c = \frac{\mu}{Z''_{\text{int}}} \frac{\partial Z''_{\text{int}}}{\partial \mu} \quad (4.24)$$

Assuming the general structure (4.16), it is in general a nontrivial fact that the anomalous dimension γ_c computed from Z''_{int} may become finite in the limit $\epsilon \rightarrow 0$. The dependence on the scale μ is encoded in the equation

$$\mu \frac{\partial \lambda}{\partial \mu} = -\epsilon \lambda \quad (4.25)$$

We can then express Eq. (4.24) in the form

$$\begin{aligned} \gamma_c &= \frac{\mu}{Z''_{\text{int}}} \frac{\partial \lambda}{\partial \mu} \frac{\partial Z''_{\text{int}}}{\partial \lambda} \\ &= -\frac{1}{Z''_{\text{int}}} \lambda \sum_{i=0}^{\infty} \frac{dc_{i+1}}{d\lambda} \frac{1}{\epsilon^i} \end{aligned} \quad (4.26)$$

Alternatively, we can write the above equation as

$$\left(1 + \sum_{i=1}^{\infty} \frac{c_i}{\epsilon^i}\right) \gamma_c = -\lambda \sum_{i=0}^{\infty} \frac{dc_{i+1}}{d\lambda} \frac{1}{\epsilon^i} \quad (4.27)$$

Assuming the finiteness of the anomalous dimension in the limit $\epsilon \rightarrow 0$, we get[32]

$$\gamma_c = -\lambda \frac{dc_1}{d\lambda} \quad (4.28)$$

and the consistency conditions for the cancellation of the poles at $\epsilon = 0$

$$\frac{dc_{i+1}}{d\lambda} = c_i \frac{dc_1}{d\lambda} \quad (4.29)$$

Quite remarkably, it can be seen that the expressions in (4.18-4.22) satisfy identically the conditions (4.29). This holds for the functions $c_i(\lambda)$ which we have obtained analytically up to order λ^7 . We have been also able to compute numerically their power series

expansion up to order λ^{18} , checking that the equations (4.29) are verified order by order with the precision allowed by the calculation. This provides a very strong evidence of the renormalizability of the theory, implying that observable quantities like γ_c can be computed from renormalized correlators to obtain finite results, dependent only on the value of the physical coupling constant.

On the other hand, we note that the result for $\mathbf{\Gamma}_{c,\text{ren}}$ is drastically modified when the electron self-energy corrections are included in the calculation of the vertex. As we have already seen in Sec. 3, the main effect of the electron self-energy is to renormalize the value of the Fermi velocity v_F . At the level discussed in that section, the self-energy corrections amount to perform the replacement in the inverse of the Dirac propagator

$$\gamma_0\omega - v_F\boldsymbol{\gamma} \cdot \mathbf{p} \rightarrow \gamma_0\omega - v_F\boldsymbol{\gamma} \cdot \mathbf{p} - \Sigma_1(\mathbf{p}) \quad (4.30)$$

with $\Sigma_1(\mathbf{p})$ given by Eq. (3.7). It is then clear that the electron self-energy diagrams can be incorporated to the ladder approximation encoded in Eq. (4.9) simply by trading the constant v_F by an effective Fermi velocity

$$\tilde{v}_F(\mathbf{p}) = v_F + \frac{e_0^2}{16\pi^2} (4\pi)^{\epsilon/2} \frac{\Gamma(\frac{1}{2}\epsilon) \Gamma(\frac{1-\epsilon}{2}) \Gamma(\frac{3-\epsilon}{2})}{\Gamma(2-\epsilon)} \frac{1}{|\mathbf{p}|^\epsilon} \quad (4.31)$$

The replacement of v_F by $\tilde{v}_F(\mathbf{p})$ in the formulas has the effect of iterating in the number of self-energy diagrams inserted in the electron and hole propagators building the vertex. The electron self-energy corrections contribute therefore to supplement the ladder series previously considered, greatly improving the diagrammatic approach for the vertex $\mathbf{\Gamma}_c$.

In order to compute the renormalization factor Z''_{int} in the ladder approximation with effective Fermi velocity $\tilde{v}_F(\mathbf{p})$, it is convenient to expand the factor $1/\tilde{v}_F(\mathbf{p})$ in powers of e_0^2 inside the integral of Eq. (4.12). The vertex $\mathbf{\Gamma}_c$ still admits a solution like that in Eq. (4.10), as each order in the perturbative expansion can be represented in terms of the precedent by integrals of the type shown in Eq. (4.12). The power series in e_0^2 contains now more poles in the ϵ parameter, as a result of the divergent behavior of $\tilde{v}_F(\mathbf{p})$ in Eq. (4.31). The poles coming from the electron self-energy corrections can be however removed at once by the renormalization of the Fermi velocity

$$v_F = Z_v v_{F,\text{ren}} \quad (4.32)$$

with

$$Z_v = 1 + b_1 \frac{1}{\epsilon} \quad (4.33)$$

The coefficient needs simply to be adjusted to

$$b_1 = -\frac{e^2}{16\pi v_{F,\text{ren}}} \quad (4.34)$$

leading then to a finite $\tilde{v}_F(\mathbf{p})$ written in terms of $v_{F,\text{ren}}$.

We have again a general structure for the renormalization factor in this improved approach

$$Z''_{\text{int}} = 1 + \sum_{i=1}^{\infty} \frac{\bar{c}_i(\lambda)}{\epsilon^i} \quad (4.35)$$

The remarkable result is that, after writing the perturbative expansion for the vertex as a power series in the renormalized coupling

$$\lambda = \frac{e^2}{4\pi v_{F,\text{ren}}} \quad (4.36)$$

one needs just a simple first-order $1/\varepsilon$ term in (4.35) to get rid of all the poles in Γ_c . That is, the renormalized vertex becomes finite in the limit $\varepsilon \rightarrow 0$ with the choice

$$\bar{c}_1(\lambda) = -\frac{1}{4}\lambda \quad (4.37)$$

$$\bar{c}_i(\lambda) = 0 \quad i \geq 2 \quad (4.38)$$

The simple pole structure of Z''_{int} in the improved ladder approximation implies the result

$$Z_v = Z''_{\text{int}} \quad (4.39)$$

We may see in this relation a nontrivial link between the renormalization of the $iv_F \boldsymbol{\gamma} \cdot \boldsymbol{\nabla}$ kinetic term and that of $\boldsymbol{\gamma} \cdot \mathbf{A}$ in the action (4.1). This feature points at a symmetry that is characteristic of a gauge invariant theory, and it can be explained in our case from inspection of the diagrams contributing to the self-energy discussed in Sec. 3 and to the vertices computed in this section, as we show next.

4.3 No-renormalization of charge and current density operators

The result (4.39) implies, together with (4.6), the preservation of the gauge invariance in the renormalized action (4.1). In this respect, there are actually Ward identities that can be derived from general principles, relating different vertex functions of the theory. This is stressed for instance in Ref. [33], where it has been also emphasized the suitability of the dimensional regularization method to preserve the gauge symmetry of the theory. We show here that the two relevant identities between the electron self-energy and the vertices Γ_0 and Γ_c can be easily obtained in the framework of the present many-body approach.

The main idea is that any electron self-energy correction can be converted into a contribution to the vertex Γ_0 or Γ_c by taking the derivative with respect to the external frequency ω_k or the external momentum \mathbf{k} of the self-energy. This fact relies on the expression of the derivatives of the free Dirac propagator

$$\begin{aligned} \frac{\partial}{\partial \omega_k} \frac{1}{\gamma_0(\omega_k - \omega_p) - v_F \boldsymbol{\gamma} \cdot (\mathbf{k} - \mathbf{p})} \\ = - \frac{1}{\gamma_0(\omega_k - \omega_p) - v_F \boldsymbol{\gamma} \cdot (\mathbf{k} - \mathbf{p})} \gamma_0 \frac{1}{\gamma_0(\omega_k - \omega_p) - v_F \boldsymbol{\gamma} \cdot (\mathbf{k} - \mathbf{p})} \end{aligned} \quad (4.40)$$

$$\begin{aligned} \frac{1}{v_F} \frac{\partial}{\partial \mathbf{k}} \frac{1}{\gamma_0(\omega_k - \omega_p) - v_F \boldsymbol{\gamma} \cdot (\mathbf{k} - \mathbf{p})} \\ = \frac{1}{\gamma_0(\omega_k - \omega_p) - v_F \boldsymbol{\gamma} \cdot (\mathbf{k} - \mathbf{p})} \boldsymbol{\gamma} \frac{1}{\gamma_0(\omega_k - \omega_p) - v_F \boldsymbol{\gamma} \cdot (\mathbf{k} - \mathbf{p})} \end{aligned} \quad (4.41)$$

In any correction to the self-energy $\Sigma(\mathbf{k}, \omega_k)$, one can choose the external frequency and momentum to circulate along the fermion lines that connect the two outer vertices of the

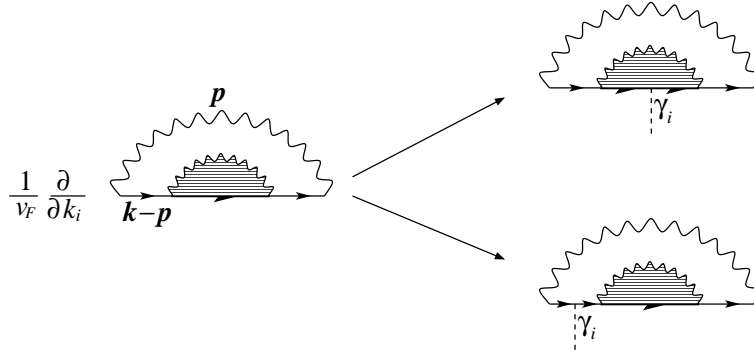


Figure 5. Schematic representation of the Ward identity between the electron self-energy and the current density vertex in the ladder approximation.

diagram. Thus, taking the derivative with respect to ω_k or \mathbf{k} implies cutting any of those internal lines in two pieces and inserting a vertex with the respective γ_0 or $\boldsymbol{\gamma}$ matrices.

The above construction becomes clear if one has in mind the diagrams building the self-consistent exchange approximation considered in Sec. 3. Taking the derivative of $\Sigma_{\text{SCEX}}(\mathbf{k})$ with respect to ω_k gives identically zero, which is consistent with the absence of corrections to Γ_0 in the ladder approximation (at zero momentum transfer). On the other hand, the derivative with respect to \mathbf{k} gives rise to two different types of diagrams contributing to Γ_c , as shown in Fig. 5. Part of them corresponds to the kind of contributions that we were considering in the ladder approximation to Γ_c without electron self-energy corrections, as seen in the upper right of the figure. But the other part of the diagrams consists of vertex corrections with self-energy insertions in the electron and hole internal lines, as illustrated in the lower right of Fig. 5. This shows that the differentiation of $\Sigma_{\text{SCEX}}(\mathbf{k})$ generates actually the whole set of vertex corrections in the ladder approximation supplemented with electron self-energy corrections.

We can then write a Ward identity of the form

$$\frac{1}{v_F} \frac{\partial}{\partial \mathbf{k}} (v_F \boldsymbol{\gamma} \cdot \mathbf{k} + \Sigma(\mathbf{k}, \omega_k)) = \Gamma_c(\mathbf{0}, 0; \mathbf{k}, \omega_k) \quad (4.42)$$

This identity implies that the renormalization of the vertex Γ_c is dictated by that of the Fermi velocity v_F . In this regard, the result $Z_v = Z''_{\text{int}}$ found above becomes a natural consequence of Eq. (4.42). Alternatively, these findings also stress the fact that the electron self-energy corrections cannot be neglected in a consistent approximation to the many-body theory of graphene, as they play a crucial role to build a gauge invariant effective action with the structure given by Eq. (4.1).

5 Staggered (sublattice odd) charge density correlations

We may also consider the renormalization of the staggered charge density operator anti-symmetric under the exchange of the two sublattices of the graphene honeycomb lattice

$$\rho_3(\mathbf{r}) = \bar{\psi}_i(\mathbf{r})\psi_i(\mathbf{r}) \quad (5.1)$$

We will define the corresponding vertex by

$$\Gamma_3(\mathbf{q}, \omega_q; \mathbf{k}, \omega_k) = \langle \rho_3(\mathbf{q}, \omega_q) \psi_i(\mathbf{k} + \mathbf{q}, \omega_k + \omega_q) \bar{\psi}_i(\mathbf{k}, \omega_k) \rangle_{\text{1PI}} \quad (5.2)$$

where 1PI denotes again that we take the irreducible part of the correlator.

The vertex Γ_3 has a clear physical significance as it enters in the correlations of the staggered charge $\rho_3(\mathbf{r})$, which is the order parameter for chiral symmetry breaking in the many-body theory. A nonvanishing expectation value $\langle \rho_3(\mathbf{r}) \rangle \neq 0$ is the signal that a mass is dynamically generated for the Dirac fermions. This means that their hamiltonian gets effectively a term of the form

$$m \int d^2r \bar{\psi}_i(\mathbf{r})\psi_i(\mathbf{r}) \quad (5.3)$$

With the mass term, the conduction and valence bands loose the perfect conical shape about the charge neutrality point, and a gap opens in the electronic spectrum. This trend of symmetry breaking is similar to that discussed long ago in Quantum Electrodynamics in two spatial dimensions[34–38]. In the present context, the dynamical mass generation is also driven by the interaction, in such a way that the condensation of $\rho_3(\mathbf{r})$ may proceed depending on the value of the coupling $e^2/4\pi v_F$ (and also on the number of fermion flavors, in a theory with a number N of different fermion species).

5.1 Staggered charge density vertex in ladder approximation

We deal first with the vertex Γ_3 in the ladder approximation, which is given again by the self-consistent equation represented diagrammatically in Fig. 4. As we are mainly interested in the cutoff dependence of the vertex, we can take in particular momentum transfer $\mathbf{q} = 0$ and $\omega_q = 0$. Given that Γ_3 must be anyhow proportional to the identity matrix, we have

$$\gamma_0 \frac{-\gamma_0 \omega_p + v_F \boldsymbol{\gamma} \cdot \mathbf{p}}{-\omega_p^2 + v_F^2 \mathbf{p}^2 - i\eta} \Gamma_3(\mathbf{0}, 0; \mathbf{p}, \omega_p) \frac{-\gamma_0 \omega_p + v_F \boldsymbol{\gamma} \cdot \mathbf{p}}{-\omega_p^2 + v_F^2 \mathbf{p}^2 - i\eta} \gamma_0 = -\frac{\Gamma_3(\mathbf{0}, 0; \mathbf{p}, \omega_p)}{-\omega_p^2 + v_F^2 \mathbf{p}^2 - i\eta} \quad (5.4)$$

Thus, the self-consistent equation for the vertex in the ladder approximation becomes

$$\Gamma_3(\mathbf{0}, 0; \mathbf{k}, \omega_k) = 1 - i \frac{e_0^2}{2} \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_p}{2\pi} \Gamma_3(\mathbf{0}, 0; \mathbf{p}, \omega_p) \frac{1}{-\omega_p^2 + v_F^2 \mathbf{p}^2 - i\eta} \frac{1}{|\mathbf{k} - \mathbf{p}|} \quad (5.5)$$

Eq. (5.5) can be further simplified by noticing that the solution cannot depend on the frequency ω_k . We end up then with the equation

$$\Gamma_3(\mathbf{0}, 0; \mathbf{k}, \omega_k) = 1 + \frac{e_0^2}{4} \int \frac{d^D p}{(2\pi)^D} \Gamma_3(\mathbf{0}, 0; \mathbf{p}, \omega_k) \frac{1}{v_F |\mathbf{p}|} \frac{1}{|\mathbf{k} - \mathbf{p}|} \quad (5.6)$$

From dimensional arguments, the solution of Eq. (5.6) can be expressed in the form

$$\Gamma_3(\mathbf{0}, 0; \mathbf{k}, \omega_k) = 1 + \sum_{n=1}^{\infty} \lambda_0^n \frac{s_n}{|\mathbf{k}|^{n\epsilon}} \quad (5.7)$$

with $\lambda_0 = e_0^2/4\pi v_F$. Each term in the series (5.7) can be obtained from the previous one, noticing that if we insert the general term in the integral at the right-hand-side of Eq. (5.6) we get

$$\frac{e_0^2}{4} \int \frac{d^D p}{(2\pi)^D} \frac{1}{|\mathbf{p}|^{m\epsilon}} \frac{1}{v_F |\mathbf{p}|} \frac{1}{|\mathbf{k} - \mathbf{p}|} = \lambda_0 \frac{(4\pi)^{\epsilon/2} \Gamma\left(\frac{m+1}{2}\epsilon\right) \Gamma\left(\frac{1-(m+1)\epsilon}{2}\right) \Gamma\left(\frac{1-\epsilon}{2}\right)}{4 \sqrt{\pi} \Gamma\left(\frac{1+m\epsilon}{2}\right) \Gamma\left(1 - \frac{m+2}{2}\epsilon\right)} \frac{1}{|\mathbf{k}|^{(m+1)\epsilon}} \quad (5.8)$$

We observe that the result of the integral diverges in the limit $\epsilon \rightarrow 0$, leading to a sequence of higher-order poles in the ϵ parameter as we look at higher perturbative levels in the solution (5.7).

The poles that appear in the computation of Γ_3 at $\epsilon = 0$ must be reabsorbed by a suitable redefinition in the scale of the operator $\rho_3(\mathbf{r})$. This is a composite field, susceptible of being renormalized by a factor Z_m which is independent of the renormalization of the elementary fields in the action[39]. This redefinition of $\rho_3(\mathbf{r})$ translates into the multiplicative renormalization of the vertex

$$\Gamma_{3,\text{ren}} = Z_m \Gamma_3 \quad (5.9)$$

The general structure of Z_m must be

$$Z_m = 1 + \sum_{i=1}^{\infty} \frac{d_i(\lambda)}{\epsilon^i} \quad (5.10)$$

in order to absorb all the poles generated by the recurrence relation (5.8).

A nontrivial check of the renormalizability of the theory is that the vertex $\Gamma_{3,\text{ren}}$ must have a finite limit as $\epsilon \rightarrow 0$, after making an appropriate choice of functions $d_i(\lambda)$ depending only on the coupling constant λ . We have seen that this is the case up to the order λ^{24} we have pursued the numerical calculation of Γ_3 , finding a set of $d_i(\lambda)$ that do not depend on the momentum \mathbf{k} of the vertex. The perturbative expansion of the functions can be

computed analytically with some effort up to order λ^8 , leading to

$$d_1(\lambda) = -\frac{1}{2}\lambda - \frac{1}{4}\log(2)\lambda^2 - \frac{1}{4}\log^2(2)\lambda^3 - \frac{128\log^3(2) + 3\zeta(3)}{384}\lambda^4 \\ - \frac{50\log^4(2) + 3\log(2)\zeta(3)}{96}\lambda^5 - \frac{4608\log^5(2) + 480\log^2(2)\zeta(3) + 5\zeta(5)}{5120}\lambda^6 \\ + \dots \quad (5.11)$$

$$d_2(\lambda) = \frac{1}{8}\lambda^2 + \frac{1}{8}\log(2)\lambda^3 + \frac{5}{32}\log^2(2)\lambda^4 + \frac{176\log^3(2) + 3\zeta(3)}{768}\lambda^5 \\ + \frac{192\log^4(2) + 9\log(2)\zeta(3)}{512}\lambda^6 + \dots \quad (5.12)$$

$$d_3(\lambda) = -\frac{1}{48}\lambda^3 - \frac{1}{32}\log(2)\lambda^4 - \frac{3}{64}\log^2(2)\lambda^5 - \frac{464\log^3(2) + 6\zeta(3)}{6144}\lambda^6 + \dots \quad (5.13)$$

$$d_4(\lambda) = \frac{1}{384}\lambda^4 + \frac{1}{192}\log(2)\lambda^5 + \frac{7}{768}\log^2(2)\lambda^6 + \dots \quad (5.14)$$

$$d_5(\lambda) = -\frac{1}{3840}\lambda^5 - \frac{1}{1536}\log(2)\lambda^6 + \dots \quad (5.15)$$

$$d_6(\lambda) = \frac{1}{46080}\lambda^6 + \dots \quad (5.16)$$

An internal consistency check of the renormalizable theory is that, as in the case of the vertex $\Gamma_{c,\text{ren}}$, the computation of physical observables like the anomalous dimension γ_m of the operator ρ_3 has to provide a finite result in the limit $\epsilon \rightarrow 0$. γ_m is defined in terms of the anomalous scaling of the vertex as a function of the dimensionful parameter μ in the renormalized theory,

$$\Gamma_{3,\text{ren}} \sim \mu^{\gamma_m} \quad (5.17)$$

The dependence on μ arises from the renormalization factor Z_m , so that

$$\gamma_m = \frac{\mu}{Z_m} \frac{\partial Z_m}{\partial \mu} \quad (5.18)$$

We can follow the same derivation as in Eqs. (4.25)-(4.27), ending up in the equation

$$\left(1 + \sum_{i=1}^{\infty} \frac{d_i(\lambda)}{\epsilon^i}\right) \gamma_m = -\lambda \sum_{i=0}^{\infty} \frac{d}{d\lambda} d_{i+1}(\lambda) \frac{1}{\epsilon^i} \quad (5.19)$$

From Eq. (5.19) we obtain a finite answer at $\epsilon = 0$ for the anomalous dimension[32]

$$\gamma_m = -\lambda \frac{d}{d\lambda} d_1(\lambda) \quad (5.20)$$

provided that the recurrence relations

$$\frac{d}{d\lambda} d_{i+1}(\lambda) = d_i(\lambda) \frac{d}{d\lambda} d_1(\lambda) \quad (5.21)$$

are identically satisfied.

We have verified that the conditions (5.21) are fulfilled, up to the order λ^{24} we have computed numerically the perturbative expansion of the functions $d_i(\lambda)$. This means that

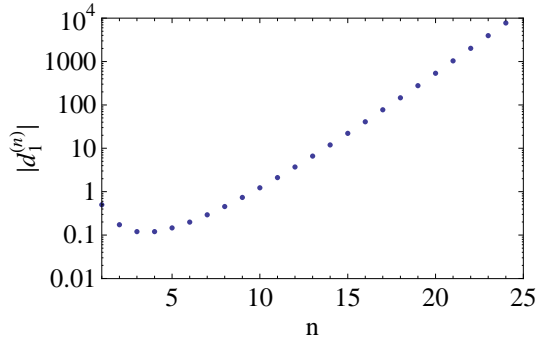


Figure 6. Plot of the absolute value of the coefficients $d_1^{(n)}$ in the expansion of $d_1(\lambda)$ as a power series of the renormalized coupling λ .

the anomalous dimension γ_m is perfectly well-defined by Eq. (5.20) in the present framework. Regarding the actual calculation, we have found that the perturbative expansion

$$d_1(\lambda) = \sum_{n=1}^{\infty} d_1^{(n)} \lambda^n \quad (5.22)$$

behaves as a power series with a finite radius of convergence λ_c . The geometric growth of the coefficients $d_1^{(n)}$ is illustrated in the plot of Fig. 6. The radius of convergence can be obtained by computing the ratio between consecutive orders of $d_1^{(n)}$, and noticing that it converges towards a limit value d . It then turns out that

$$-d_1(\lambda) = \sum_{n=1}^{\infty} d^n \lambda^n + \text{regular terms} \quad (5.23)$$

An excellent fit of the n -dependence of $d_1^{(n+1)}/d_1^{(n)}$ is achieved by assuming the scaling behavior

$$\frac{d_1^{(n)}}{d_1^{(n-1)}} = d + \frac{d'}{n} + \frac{d''}{n^2} + \frac{d'''}{n^3} + \dots \quad (5.24)$$

We obtain in this way an estimate of the radius of convergence

$$\lambda_c = \frac{1}{d} \approx 0.456947 \quad (5.25)$$

The singular behavior of the anomalous dimension γ_m is the manifestation of the divergence of the vertex $\Gamma_{3,\text{ren}}$ at the critical coupling λ_c . It also implies the divergence of the correlators of the staggered charge density $\rho_3(\mathbf{r})$ as these need to be multiplicatively renormalized by factors of Z_m , which means that their anomalous dimensions are given by multiples of γ_m . What we find therefore at the critical point λ_c is the signature of the dynamical breakdown of the chiral symmetry in the Dirac theory. The predicted critical coupling in Eq. (5.25) turns out to be a very accurate approximation to the critical value

$$\lambda_c = \frac{8\pi^2}{(\Gamma(\frac{1}{4}))^4} \quad (5.26)$$

which has been obtained by Gamayun *et al.* in Ref. [27] by a quite different approach, consisting in the self-consistent resolution of the gap equation for the Dirac fermions (for another connection between the branching point of the gap equation and the singularity in the vertex, see also Ref. [40]). This remarkable coincidence between the results of two completely different methodologies can be taken as the reflection that they are encoding at the end an equivalent sum of many-body corrections, thus providing a nice check of the reliability of our computational framework based on the ladder approximation to vertex functions.

5.2 Staggered charge density vertex supplemented with self-energy corrections

We have anyhow to keep in mind that the electron self-energy corrections need to be incorporated to reach sensible results for $\Gamma_{3,\text{ren}}$, as we learned from the renormalization of the current density vertex. In the present case, the ladder approximation can be also improved by inserting the series of electron self-energy diagrams in the internal electron and hole states of the vertex. From a computational point of view, this can be achieved by replacing the constant v_F in the integrand of Eq. (5.6) by the effective Fermi velocity $\tilde{v}_F(\mathbf{p})$ dressed with the self-energy corrections in Eq. (4.31). It can be easily seen that expanding the latter in powers of e_0^2 corresponds to generating the iteration of self-energy corrections to the internal fermion propagators in the equation of Fig. 4.

A solution of the form (5.7) can be still found for $\Gamma_3(\mathbf{0}, 0; \mathbf{k}, \omega_k)$, where now each term in the series can be obtained from all the precedent by expanding $\tilde{v}_F(\mathbf{p})$ in powers of e_0^2 in Eq. (5.6) and using repeatedly the formula (5.8). The Fermi velocity v_F needs to be renormalized to absorb the divergence of $\tilde{v}_F(\mathbf{p})$ in the limit $\epsilon \rightarrow 0$, for which we define

$$v_F = Z_v v_{F,\text{ren}} \quad (5.27)$$

As in subsection 4.2, Z_v just contains a simple pole

$$Z_v = 1 + b_1 \frac{1}{\epsilon} \quad (5.28)$$

with $b_1 = -e^2/16\pi v_{F,\text{ren}}$. After subtraction of the self-energy pole, the rest of poles in Γ_3 must be reabsorbed by the multiplicative renormalization (5.9), where now Z_m is given by a different series

$$Z_m = 1 + \sum_{i=1}^{\infty} \frac{\tilde{d}_i(\lambda)}{\epsilon^i} \quad (5.29)$$

One can see that in this case $\Gamma_{3,\text{ren}}$ can be made also finite in the limit $\epsilon \rightarrow 0$, with a set of functions $\tilde{d}_i(\lambda)$ that only depend on the renormalized coupling constant

$$\lambda = \mu^{-\epsilon} Z_v \lambda_0 = \frac{e^2}{4\pi v_{F,\text{ren}}} \quad (5.30)$$

The first orders in the perturbative expansion are given for instance by

$$\begin{aligned}\tilde{d}_1(\lambda) = & -\frac{1}{2}\lambda - \frac{1}{8}\log(2)\lambda^2 - \frac{\pi^2 + 120\log^2(2)}{1152}\lambda^3 - \frac{10\pi^2\log(2) + 688\log^3(2) + 15\zeta(3)}{6144}\lambda^4 \\ & - \frac{13\pi^4 + 2064\pi^2\log^2(2) + 144(716\log^4(2) + 37\log(2)\zeta(3))}{737280}\lambda^5 \\ & - \frac{1}{2949120}(169\pi^4\log(2) + 567744\log^5(2) + 49320\log^2(2)\zeta(3) \\ & + 5\pi^2(2864\log^3(2) + 37\zeta(3)) + 1125\zeta(5))\lambda^6 + \dots\end{aligned}\quad (5.31)$$

$$\begin{aligned}\tilde{d}_2(\lambda) = & \frac{1}{16}\lambda^2 + \frac{1}{24}\log(2)\lambda^3 + \frac{5\pi^2 + 744\log^2(2)}{18432}\lambda^4 \\ & + \frac{110\pi^2\log(2) + 8592\log^3(2) + 135\zeta(3)}{184320}\lambda^5 \\ & + \frac{293\pi^4 + 58944\pi^2\log^2(2) + 72(44392\log^4(2) + 1779\log(2)\zeta(3))}{53084160}\lambda^6 + \dots\end{aligned}\quad (5.32)$$

$$\begin{aligned}\tilde{d}_3(\lambda) = & -\frac{1}{768}\log(2)\lambda^4 - \frac{\pi^2 + 360\log^2(2)}{184320}\lambda^5 \\ & - \frac{100\pi^2\log(2) + 11904\log^3(2) + 45\zeta(3)}{4423680}\lambda^6 + \dots\end{aligned}\quad (5.33)$$

$$\tilde{d}_4(\lambda) = -\frac{1}{7680}\log(2)\lambda^5 - \frac{\pi^2 + 280\log^2(2)}{1474560}\lambda^6 + \dots\quad (5.34)$$

$$\tilde{d}_5(\lambda) = -\frac{1}{61440}\log(2)\lambda^6 + \dots\quad (5.35)$$

We have computed the expansions of the functions $\tilde{d}_i(\lambda)$ numerically up to order λ^{24} , checking that the coefficients do not depend on the momentum \mathbf{k} of the vertex. This is the essential requirement guaranteeing the renormalizability of the theory, by which the divergences in the limit $\epsilon \rightarrow 0$ can be absorbed into the redefinition of a finite number of local operators.

We can proceed to the computation of the anomalous dimension γ_m in the present approach, taking again into account that the scale dependence of $\Gamma_{3,\text{ren}}$ stems from Z_m , so that

$$\gamma_m = \frac{\mu}{Z_m} \frac{\partial \lambda}{\partial \mu} \frac{\partial Z_m}{\partial \lambda}\quad (5.36)$$

The dependence of the renormalized coupling λ on μ can be obtained by differentiating (5.30), leading to

$$\mu \frac{\partial \lambda}{\partial \mu} = -\epsilon \lambda + \frac{\lambda}{Z_v} \mu \frac{\partial \lambda}{\partial \mu} \frac{\partial Z_v}{\partial \lambda}\quad (5.37)$$

Using the fact that $Z_v = 1 + \lambda(\partial Z_v / \partial \lambda)$, we obtain

$$\begin{aligned}\mu \frac{\partial \lambda}{\partial \mu} &= -\epsilon \lambda Z_v \\ &= -\epsilon \lambda - \lambda b_1(\lambda)\end{aligned}\quad (5.38)$$

This result can be now introduced in Eq. (5.36), finding that

$$\gamma_m = \frac{1}{Z_m}(-\epsilon \lambda - \lambda b_1(\lambda)) \sum_{i=1}^{\infty} \frac{1}{\epsilon^i} \frac{d}{d\lambda} \tilde{d}_i(\lambda)\quad (5.39)$$

Equivalently, we can write

$$\left(1 + \sum_{i=1}^{\infty} \frac{\tilde{d}_i(\lambda)}{\epsilon^i}\right) \gamma_m = -\lambda \sum_{i=0}^{\infty} \frac{1}{\epsilon^i} \frac{d}{d\lambda} \tilde{d}_{i+1}(\lambda) - \lambda b_1(\lambda) \sum_{i=1}^{\infty} \frac{1}{\epsilon^i} \frac{d}{d\lambda} \tilde{d}_i(\lambda) \quad (5.40)$$

Assuming that γ_m must have a finite limit as $\epsilon \rightarrow 0$, we get

$$\gamma_m = -\lambda \frac{d}{d\lambda} \tilde{d}_1(\lambda) \quad (5.41)$$

together with the conditions for the cancellation of all the pole contributions to γ_m [32]

$$\tilde{d}_i(\lambda) \frac{d}{d\lambda} \tilde{d}_1(\lambda) = \frac{d}{d\lambda} \tilde{d}_{i+1}(\lambda) + b_1(\lambda) \frac{d}{d\lambda} \tilde{d}_i(\lambda) \quad (5.42)$$

Quite remarkably, we have verified that the conditions (5.42) are indeed satisfied by the perturbative series of the functions $\tilde{d}_i(\lambda)$, at least up to the order λ^{24} we have been able to carry out numerically the expansion. This guarantees that Eq. (5.41) can be used to obtain a finite result for γ_m , only dependent on the value of the coupling constant λ . Computing the perturbative expansion

$$\tilde{d}_1(\lambda) = \sum_{n=1}^{\infty} \tilde{d}_1^{(n)} \lambda^n \quad (5.43)$$

we have checked that in this case again the coefficients $\tilde{d}_1^{(n)}$ grow geometrically with the order n , as shown in Fig. 7. The ratio $\tilde{d}_1^{(n+1)}/\tilde{d}_1^{(n)}$ converges to a limit value \tilde{d} , in such a way that

$$-\tilde{d}_1(\lambda) = \sum_{n=1}^{\infty} \tilde{d}^n \lambda^n + \text{regular terms} \quad (5.44)$$

We have found that the points $\tilde{d}_1^{(n+1)}/\tilde{d}_1^{(n)}$ can be fitted quite accurately by the scaling behavior

$$\frac{\tilde{d}_1^{(n)}}{\tilde{d}_1^{(n-1)}} = \tilde{d} + \frac{\tilde{d}'}{n} + \frac{\tilde{d}''}{n^2} + \frac{\tilde{d}'''}{n^3} + \dots \quad (5.45)$$

We obtain in this way a finite radius of convergence for the perturbative expansion

$$\lambda_c = \frac{1}{\tilde{d}} \approx 0.544775 \quad (5.46)$$

where the error is estimated to be in the last digit (in similar fashion as for the critical value given in (5.25). This represents a considerable gain in precision with respect to the calculation reported in Ref. [31], which is now made possible by our ability to carry out the power series expansion for $\tilde{d}_1(\lambda)$ up to seven orders beyond that achieved in that paper.

The existence of a critical coupling λ_c implies that the many-body theory of interacting Dirac fermions enters a new phase for sufficiently large strength of the Coulomb interaction. According to our preceding discussion, this phase is characterized by the condensation of the staggered charge density, which signals the dynamical breakdown of the chiral symmetry of the theory. We find therefore that this phenomenon takes place even after accounting

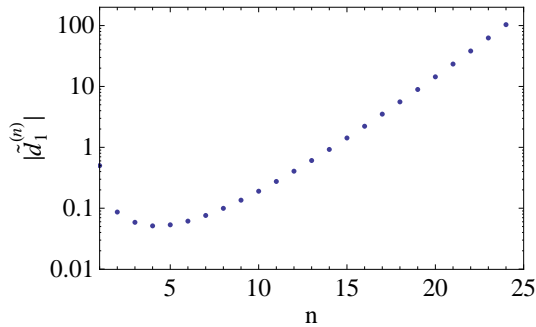


Figure 7. Plot of the absolute value of the coefficients $\tilde{d}_1^{(n)}$ in the expansion of $\tilde{d}_1(\lambda)$ as a power series of the renormalized coupling λ .

for the electron self-energy corrections in the many-body theory. Thus, the effect of Fermi velocity renormalization tends to reduce the effective interaction strength, but it does not prevent the dynamical mass generation, leading instead to a larger value (5.46) of the critical coupling in comparison to the approach without switching on the electron self-energy corrections (see also Ref. [41]).

6 Conclusions

The above results provide strong evidence that the model of interacting Dirac fermions in graphene constitutes a completely renormalizable field theory, in the sense that all the cutoff dependences can be absorbed into the redefinition of a finite number of local operators. We have proven this fact for different vertex functions in the ladder approximation, as well as when this is supplemented by electron self-energy corrections to the electron and hole states in the vertices. We have seen that there is a nontrivial cancellation of poles in the dimensionally regularized anomalous dimensions of different fermion bilinears, a feature typically enforced by nonperturbative equations for the residues of the poles in renormalizable quantum field theories. It would be then interesting to pursue the program of renormalization in other approximations with higher diagrammatic content, testing the cutoff independence of anomalous exponents in the graphene electron system.

The other important conclusion we reach is that the incorporation of electron self-energy corrections is in general required to preserve the gauge invariance that relates the kinetic and the interaction terms in the effective action of the theory. When looking at corrections to the current density vertex in the ladder approximation, gauge invariance implies in particular the complete cancellation between the cutoff-dependent part of self-energy and vertex corrections to all orders in the perturbation expansion, as we have checked explicitly in our calculation with dimensional regularization.

From a practical point of view, our computational approach has allowed us to address the question of the chiral symmetry breaking in the interacting theory of Dirac fermions in graphene. Making use of the renormalizability of the theory, we have shown that quantities

like the anomalous dimension of the staggered charge density operator can be determined in terms of the renormalized coupling alone, allowing to characterize the dynamical symmetry breaking at the critical coupling given by the finite radius of convergence of the perturbative expansion.

We note anyhow that the critical value found for the coupling λ cannot be used directly to predict the onset of dynamical symmetry breaking in real graphene, as the e - e interaction is affected in general by screening processes that may reduce significantly its strength. This means that the critical coupling λ_c we have computed should be referred actually to the effective interaction strength after incorporating screening corrections. These depend in general on intrinsic factors, like the number N of different fermion flavors that enter in the polarization of the system. Under static RPA screening, for instance, the coupling λ of the effective interaction is related to the bare fine structure constant α in graphene by the expression

$$\lambda = \frac{\alpha}{1 + \frac{N\pi}{4}\alpha} \quad (6.1)$$

N being the number of four-component Dirac fermions. In the physical case $N = 2$, the nominal coupling of graphene in vacuum $\alpha \approx 2.2$ leads to the estimate $\lambda \approx 0.49$, which is above the critical coupling (5.25) obtained in the pure ladder approximation, but below the value (5.46) found after incorporating the electron self-energy corrections. The latter critical value of λ translates therefore into a more stringent bound on the nominal value of e^2/v_F for the development of chiral symmetry breaking. It is worth noticing however that the more sensible screening approaches, considering the dynamical polarization of electron-hole pairs, still predict the dynamical mass generation for the coupling $\alpha \approx 2.2$ corresponding to graphene isolated in vacuum[29, 31].

The reliability of our renormalization procedure is reinforced by the fact that the critical coupling we obtain from the sum of ladder diagrams for the vertex matches with great accuracy the value found within a quite different approach to chiral symmetry breaking in graphene, based on the self-consistent resolution of the gap equation for the Dirac fermions[27]. This provides good motivation to check whether a similar agreement can be reached after incorporating the electron self-energy corrections in the self-consistent gap equation, as well as to extend by any other feasible means the approach devised in the present paper.

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